to assume that the method of Adamson and Nicholls for the prediction of the Mach disk location in plumes formed in still air would apply with similar accuracy for exhaust into a co-flowing airstream. Finally, since the axial Mach number distribution upstream of the Mach disk is independent of the external condition, we conclude from these arguments that the axial location of the Mach disk in a plume contained in a co-flowing airstream would, for the same exit-to-ambient pressure ratio, be identical to its location in a plume formed in still air (the limitations of this conclusion will be discussed later).

To test the validity of the conclusion reached, Mach disk locations were extracted from Schlieren photographs of plumes formed in the presence of supersonic external flows.[5,10] These were then compared with the results of the experiments for exhaust into still air as represented by Lewis and Carlson's empirical correlation. The result, presented in Fig. 1, supports the conclusion that, over the range of conditions investigated, the axial position of the Mach disk is essentially independent of the Mach number in the external flow.

Limitations on Conclusion

1) From the results presented, the height of the nozzle base appears to have no influence on the Mach disk location. Such would not be true in a situation where the first plume wavelength were totally immersed in the base flow. Under this condition, the Mach disk would be found at a location corresponding to a pressure ratio equal to the exit-to-base pressure ratio. The latter represents, however, an infrequently encountered flow situation. 2) It cannot be assumed that a Mach disk, if found in a plume in still air, will always be found in the presence of external flow. While the external flow has no influence on the axial Mach number distribution in the region previously described, it does have an influence on the trajectory of the intercepting shock wave. Thus, at a fixed pressure ratio, the intercepting shock is displaced toward the axis of symmetry as the freestream Mach number is increased. If the freestream Mach number is sufficiently high, the intercepting shock can reach the axis upstream of the point predicted for the Mach disk location. Under this condition, a Mach disk will not form, and a regular reflection will occur.

References

[1] Victor, A. C. and Buecher, R. W., "An Analytical Approach to the Turbulent Mixing of Coaxial Jets," Naval Weapons Center, China Lake, Calif., 1966.

[2] Smoot, L. D. and Purcell, W. E., "Model for Mixing of a Compressible Free Jet with a Moving Environment," *AIAA Journal*, Vol. 5, No. 11, Nov. 1967, pp. 2049–2052.

[3] Mikatarian, R. R. and Pergament, H. S., "Aerochem Axisymmetric Mixing with Nonequilibrium Chemistry Computer Program," Aero-Chem TP-200, Aerochem Research Labs., Princeton, N.J., 1969.

[4] Buckley, F. T. and Myers, R. B., "Studies of Solid Rocket Exhaust Afterburning with a Finite-Difference Equilibrium Chemistry Plume Model," *Proceedings JANNAF 6th Plume Technology Conference*, CPIA-209, Chemical Propulsion Information Agency, The Johns Hopkins Univ., Silver Spring, Md., 1971, pp. 59–79.

[5] Love, E. S., Grigsby, C. E., Lee, L. P., and Woodling, M. J., "Experimental and Theoretical Studies of Axisymmetric Free Jets," TR R-6, 1959, NASA.

[6] Adamson, T. C. and Nicholls, J. A., "On the Structure of Jets from Highly Underexpanded Nozzles into Still Air," *Journal of Aeronautical Sciences*, Vol. 26, No. 1, pp. 16–24, Jan. 1959.

[7] Owen, P. L. and Thornhill, C. K., "The Flow in an Axially Symmetric Supersonic Jet from a Nearby Sonic Orifice into a Vacuum," ARC-TR R & M 2616, 1950, British Aeronautical Research Council, London.

[8] Lewis, C. H. and Carlson, D. J., "Normal Shock Location in Underexpanded Gas and Gas-Particle Jets," *AIAA Journal*, Vol. 2, No. 4, pp. 776–777, April 1964.

[9] Furby, B. E., "Studies of an Air Jet from a Choke and Their Application to Afterburning of a Rocket Jet," TN-PD 46, March 1961, Dept. of Supply, Australian Defense Scientific Service, Weapons Research Establishment, Melbourne, Australia.

[10] Rousso, M. D. and Baughman, L. E., "Spreading Characteristics of a Jet Expanding from Chocked Nozzles at Mach 1.91," TN 3836, 1956, NACA.

Influence of Subsonic Potential Flow on the Buckling of Thin Panels under Edge Compression

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THIS Note was prompted by recent publications[1,2] dealing with the title problem on the basis of ad-hoc approximate aerodynamic theories of a rather intuitive character. Here, the aerodynamic operator is derived from the linearized theory of subsonic potential flow and the solution is obtained by Galerkin's method.

Consider a thin elastic panel of infinite width simply supported along its edges $\bar{x} = 0$ and $\bar{x} = l$ and loaded by uniformly distributed compressive forces N_x. The panel is set in an infinite plane $z = 0$ and placed in a fluid flow of subsonic velocity U in the $\bar{x}$ direction. To investigate the static stability of the panel (static divergence precedes the onset of flutter in subsonic flow, e.g.,[3]) we impose on it a small lateral deflection $w(x)$ thereby changing the dynamic pressure of the ambient flow by $\Delta p(x)$ and seeking the values of N_x and U at which the deflected panel is in equilibrium.

The equation of equilibrium is

$$D\frac{d^4w}{dx^4} + N_x l^2 \frac{d^2w}{dx^2} + \Delta p(x) l^4 = 0 \tag{1}$$

subject to the boundary conditions:

$$w = 0,\ w'' = 0 \quad \text{at} \quad x = 0, 1 \tag{2}$$

where x is the dimensionless streamwise coordinate ($x = \bar{x}/l$) and D the bending stiffness of the panel.

The dynamic pressure on the upper surface of the panel is given by the Bernoulli equation

$$p_u = -\frac{\rho U}{l}\frac{\partial \phi_u}{\partial x}\bigg|_{z=0^+} \tag{3}$$

where ρ is the mass density of the fluid and ϕ_u the perturbation velocity potential in the upper half space ($z \geqq 0$). For a subsonic ($M < 1$), compressible inviscid irrotational fluid, $\phi_u(x, z)$ is determined by solving the equation

$$(1-M^2)\frac{\partial^2 \phi_u}{\partial x^2} + \frac{\partial^2 \phi_u}{\partial z^2} = 0 \tag{4}$$

subject to the boundary conditions:

$$\frac{\partial \phi_u}{\partial z}\bigg|_{z=0^+} = \begin{cases} U\dfrac{dw}{dx} & \text{for} \quad 0 \leqq x \leqq 1 \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

and

$$\partial\phi_u/\partial z,\ \partial\phi_u/\partial x \to 0 \quad \text{at} \quad z \to \infty \tag{6}$$

Received May 16, 1974; revision received July 11, 1974. The author acknowledges helpful discussions with J. Golecki and M. Reihav. Thanks are due E. Jawson who competently performed the numerical computations.

Index categories: Aeroelasticity and Hydroelasticity; Nonsteady Aerodynamics; Structural Stability Analysis.

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Introducing the Fourier transforms

$$\left.\begin{aligned}\phi_u^*(\alpha, z) &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \phi_u(x, z)e^{-i\alpha x}\,dx \\ F^*(\alpha) &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \frac{dw}{dx}e^{-i\alpha x}\,dx \\ p^*(\alpha) &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} p_u(x)e^{-i\alpha x}\,dx\end{aligned}\right\} \tag{7}$$

we find that Eq. (4) is equivalent to the ordinary differential equation $d^2\phi_u^*/dz^2 - (1-M^2)\alpha^2\phi_u^* = 0$ and its solution satisfying Eqs. (5) and (6) is

$$\phi_u^*(\alpha, z) = -\frac{U}{\beta}\frac{F^*(\alpha)e^{-|\alpha|\beta z}}{|\alpha|} \tag{8}$$

where

$$\beta = (1-M^2)^{1/2}$$

Substituting Eq. (8) in the transformed Eq. (3) we have

$$p_u^*(\alpha) = i\frac{\rho U^2}{\beta l}\frac{\alpha F^*(\alpha)}{|\alpha|}e^{-|\alpha|\beta z} \quad \text{at} \quad z = 0$$

and on inversion

$$p_u(x) = i\frac{\rho U^2}{\beta l}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{dw}{d\xi}H(x-\xi, z=0)\,d\xi \tag{9}$$

where

$$H(x, z) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{\alpha}{|\alpha|}e^{-|\alpha|\beta z + i\alpha x}\,d\alpha =$$

$$i\left(\frac{2}{\pi}\right)^{1/2}\int_0^{\infty} e^{-|\alpha|\beta z}\sin\alpha x\,d\alpha = i\left(\frac{2}{\pi}\right)^{1/2}\frac{x}{x^2+\beta^2 z^2}$$

Substituting the last expression in Eq. (9), we obtain

$$p_u(x) = -\frac{\rho U^2}{\beta l}\frac{1}{\pi}\int_0^1 \frac{dw}{d\xi}\frac{d\xi}{x-\xi} \tag{10}$$

It is assumed that both sides of the panel are exposed to the flow. Proceeding in a similar way as above, it can be shown that, for the lower half-space ($z \leqq 0$), the perturbation velocity potential is

$$\phi_L^*(\alpha, z) = \frac{U}{\beta}\frac{F^*(\alpha)e^{\beta|\alpha|z}}{|\alpha|} \tag{8a}$$

when by simple transformations the dynamic pressure below the panel $p_L = -p_u$, and the total dynamic pressure acting on the panel $\Delta p = p_u - p_L$ equals

$$\Delta p(x) = 2p_u = -\frac{2\rho U^2}{\pi\beta l}\int_0^1 \frac{dw}{d\xi}\frac{d\xi}{x-\xi} \tag{11}$$

The expression (10) is also valid for a thin symmetrical airfoil in noncirculatory flow (in this case however $p_L = +p_u$). It was derived in the wing theory by many authors (using other methods), and the integral in Eq. (10) was thoroughly investigated (e.g. [4,5]). To evaluate this improper integral its principal value should be taken, and its singularities at the leading and trailing edges (which vanish only in the particular case when the panel is clamped along both edges) have a negligible effect. These integrable singularities entail no mathematical difficulties in the subsequent calculations.

Introducing dimensionless coefficients

$$\lambda^2 = \rho U^2 l^3/D(1-M^2)^{1/2}, \qquad s = N_x l^2/\pi^2 D = N_x/N_{EO}$$

where N_{EO} is the critical ("Euler") force for the panel in a fluid at rest, and substituting Eq. (11) in Eq. (1) we have

$$\frac{d^4w}{dx^4} + \pi^2 s\frac{d^2w}{dx^2} - 2\frac{\lambda^2}{\pi}\int_0^1 \frac{dw}{d\xi}\frac{d\xi}{x-\xi} = 0 \tag{12}$$

Equation (12) is discussed in Ref. 6 where the aerodynamic operator Eq. (10) was adopted without derivation, from the theory of thin airfoils. Following Ref. 6 we assume that

$$w(x) = \sum_{n=1}^{N} a_n \sin n\pi x \tag{13}$$

thus satisfying the boundary conditions, Eqs. (2). Substituting Eq. (13) in Eq. (12) and using Galerkin's method, a system of N algebraic equations is obtained in the N unknown coefficients a_n. The system has a nontrivial solution provided its determinant vanishes, thus yielding the characteristic equation

$$\text{Det}\,\|\pi^4(k^4 - sk^2)\delta_{nk} - 4\lambda^2 A_{nk}\| = 0 \qquad n, k = 1, 2, \ldots, N$$

where δ_{nk} is the Kronecker symbol and

$$A_{nk} = k\int_0^1 \sin n\pi x \int_0^1 \frac{\cos k\pi\xi}{x-\xi}\,d\xi\,dx \tag{14}$$

The diagonal terms A_{nn} turned out to be much larger than the nondiagonal ones, and calculations showed that the lowest critical parameters are obtainable to a high degree of accuracy using the first mode only:

$$\pi^4(1-s) - 4\lambda^2 A_{11} = 0 \tag{15}$$

where † $A_{11} = 1.2046$.

Equation (15) implies that in the absence of compressive forces ($s \equiv 0$), divergence of the panel occurs at the critical flow velocity U_{DO} defined by

$$\lambda_{DO}^2 = \pi^4/4A_{11} = 20.23 \tag{16}$$

At lower flow velocities Eq. (15) can be rewritten in the following symmetrical form

$$N_{cr} = N_{EO}[1 - (\lambda^2/\lambda_{DO}^2)] \qquad \lambda^2 \leqq \lambda_{DO}^2 \tag{17}$$

or explicitly

$$N_{cr} = N_{EO} - (1/2.05)(\rho U^2 l)/(1-M^2)^{1/2} \tag{18}$$

Extreme caution must be exercised in extending the simple formula (17) to other configurations, especially to panels with finite aspect ratios. Systematic numerical calculations are necessary to ascertain how far such extension is permissible.

References

[1] Roth, W., "Analogy between Buckling and Flutter of Plates" (in German), *Acta Mechanica*, Vol. 13, No. 1, Jan. 1972, pp. 55–67.

[2] Rosemeier, G. E., "Aerodynamic Stability of Space Structures Under Compression" (in German), *Bautechnik*, Vol. 49, No. 1, Jan. 1972, pp. 7–11.

[3] Kornecki, A., "Static and Dynamic Instability of Panels and Cylindrical Shells in Subsonic Potential Flow," *Journal of Sound and Vibration*, Vol. 32, No. 1, Jan. 1974, pp. 251–263.

[4] Neumark, S., "Velocity Distribution on Straight and Sweptback Wings," R & M 2713, 1952, British Aeronautical Research Council, Farnborough, England.

[5] Van Dyke, M. D., "Second-Order Subsonic Airfoil Theory Including Edge Effect," Rept. 1274, 1956, National Advisory Committee for Aeronautics.

[6] Ishii, T., "Aeroelastic Instabilities of Simply Supported Panels in Subsonic Flow," AIAA Paper 65-772, Los Angeles, Calif., 1965.

† Numerical results obtained (up to $N = 6$) are somewhat different from those listed in Ref. 6.

Square-Root Variable Metric Method for Function Minimization

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Introduction

ONE of the best known methods of minimizing a function of n variables is classified as a variable metric method.[1,2] Probably the best known of these methods is Davidon's method.

Received June 10, 1974.

Index categories: Navigation, Control, and Guidance Theory; Spacecraft Navigation, Guidance, and Flight-Path Control Systems.

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